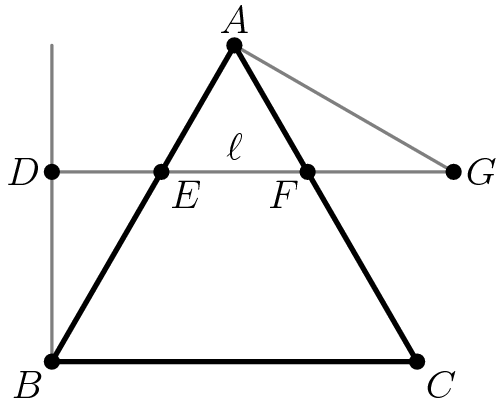
# 2021 AIME II Problems & Solutions

1. 2021 AIME II Problem 1

* Find the arithmetic mean of all the three-digit palindromes. (Recall that a palindrome is a number that reads the same forward and backward, such as or .)
* Solution 1
* Recall that the arithmetic mean of all the digit palindromes is just the average of the largest and smallest digit palindromes, and in this case the palindromes are and and which is the final answer.
* Solution 2
* For any palindrome note that is The average for is since can be any of or The average for is since is either or Therefore, the answer is

1. 2021 AIME II Problem 2

* Equilateral triangle has side length . Point lies on the same side of line as such that . The line through parallel to line intersects sides and at points and , respectively. Point lies on such that is between and , is isosceles, and the ratio of the area of to the area of is . Find .
* 
* Solution (Area Formulas for Triangles)
* By angle chasing, we conclude that is a triangle, and is a triangle.
* Let It follows that and By the side-length ratios in we have and
* Let the brackets denote areas. We have and
* We set up and solve an equation for
* Since it is clear that Therefore, we take the positive square root for both sides:

1. 2021 AIME II Problem 3

* Find the number of permutations of numbers such that the sum of five products is divisible by
* Solution
* Since is one of the numbers, a product with a in it is automatically divisible by so WLOG we will multiply by afterward since any of would be after some cancelation we see that now all we need to find is the number of ways that is divisible by since is never divisible by now we just need to find the number of ways is divisible by Note that and can be or We have ways to designate and for a total of So the desired answer is
* Solution 2 (Cyclic Symmetry and Casework)
* The expression has cyclic symmetry. Without the loss of generality, let It follows that We have:
* are congruent to in some order.
* We construct the following table for the case with all values in modulo

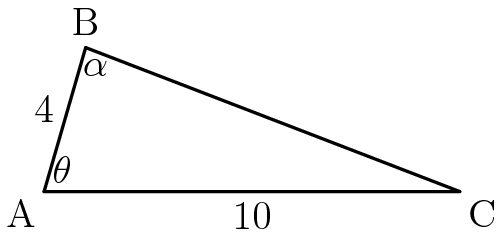
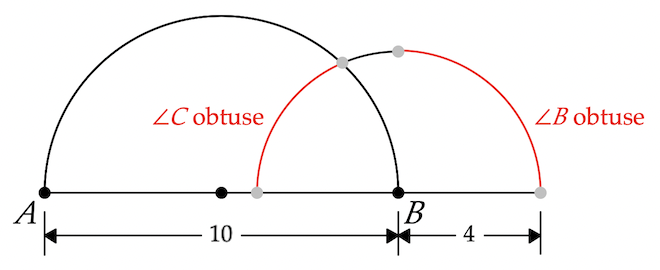
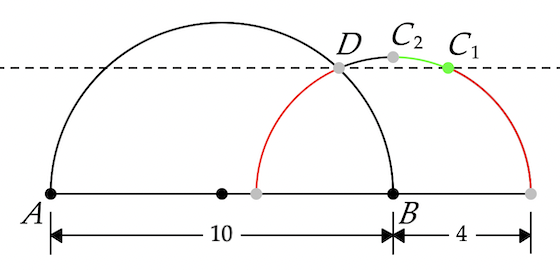
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| Row |  |  |  |  |  |  |
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* For Row 1, can be either or and can be either or By the Multiplication Principle, Row 1 produces permutations. Similarly, Rows 2, 5, and 6 each produce permutations.
* Together, we get permutations for the case By the cyclic symmetry, the cases and all have the same count. Therefore, the total number of permutations is
* Solution 3
* WLOG, let So, the terms are divisible by .
* We are left with and . We need . The only way is when They are or .
* The numbers left with us are which are respectively.
* (of or ) (of or ) = .
* (of or ) (of or ) =
* But, as we have just two and two . Hence, We will have to take and . Among these two, we have a and in common, i.e.  (because and . are common in and ).
* So, i.e.  values.
* For each value of we get values for . Hence, in total, we have ways.
* But any of the can be . So, .

1. 2021 AIME II Problem 4

* There are real numbers and such that is a root of and is a root of These two polynomials share a complex root where and are positive integers and Find
* Solution 1 (Complex Conjugate Root Theorem and Vieta’s Formulas)
* By the Complex Conjugate Root Theorem, the imaginary roots for each of and are complex conjugates. Let and It follows that the roots of are and the roots of are
* We know that
* Applying Vieta’s Formulas to we have Substituting into this equation, we get
* Applying Vieta’s Formulas to we have or Substituting and into this equation, we get
* Finally, the answer is
* Solution 2 (Somewhat Bashy)
* , hence
* Also, , hence
* satisfies both we can put it in both equations and equate to 0.
* In the first equation, we get Simplifying this further, we get
* Hence, and
* In the second equation, we get Simplifying this further, we get
* Hence, and
* Comparing (1) and (2),
* and
* ;
* Substituting these in gives,
* This simplifies to
* Hence,
* Consider case of :
* Also,
* (because c = 1) Also, Also, Equation (2) gives
* Solving (4) and (5) simultaneously gives
* [AIME can not have more than one answer, so we can stop here also 😁… Not suitable for Subjective exam]
* Hence,
* Solution 3 (Heavy Calculation Solution)
* start off by applying vieta’s and you will find that and . After that, we have to use the fact that and are roots of and , respectively. Since we know that if you substitute the root of a function back into the function, the output is zero, therefore and and you can set these two equations equal to each other while also substituting the values of , , , and above to give you , then you can rearrange the equation into . With this property, we know that is divisible by therefore that means which results in which finally gives us m=10 mod 21. We can test the first obvious value of which is and we see that this works as we get and . That means your answer will be
* Solution 4 (Synthetic Division)
* We note that and for some polynomials and .
* Through synthetic division (ignoring the remainder as we can set and to constant values such that the remainder is zero), , and .
* By the complex conjugate root theorem, we know that and share the same roots, and they share the same leading coefficient, so .
* Therefore, and . Solving the system of equations, we get and , so .
* Finally, by the quadratic formula, we have roots of , so our final answer is
* Solution 5 (Fast and Easy)
* We plug into the equation obtaining , likewise, plugging -21 into the second equation gets .
* Both equations must have 3 solutions exactly, so the other two solutions must be and .
* By Vieta’s, the sum of the roots in the first equation is , so must be .
* Next, using Vieta’s theorem on the second equation, you get: However, since we know that the sum of the roots with complex numbers are 20, we can factor out the terms with -21, so
* Given that is , then is equal to .
* Therefore, the answer to the equation is

1. 2021 AIME II Problem 5

* For positive real numbers , let denote the set of all obtuse triangles that have area and two sides with lengths and . The set of all for which is nonempty, but all triangles in are congruent, is an interval . Find .
* Solution 1
* We start by defining a triangle. The two small sides MUST add to a larger sum than the long side. We are given and as the sides, so we know that the 3rd side is between and , exclusive. We also have to consider the word OBTUSE triangles. That means that the two small sides squared is less than the 3rd side. So the triangles’ sides are between and exclusive, and the larger bound is between and , exclusive. The area of these triangles are from (straight line) to on the first “small bound” and the larger bound is between and . is our first equation, and is our 2nd equation. Therefore, the area is between and , so our final answer is .
* Solution 2 (Inequalities and Casework)
* If and are the side-lengths of an obtuse triangle with then both of the following must be satisfied:
* Triangle Inequality Theorem:
* Pythagorean Inequality Theorem:
* For one such obtuse triangle, let and be its side-lengths and be its area. We apply casework to its longest side:
* Case (1): The longest side has length so
* By the Triangle Inequality Theorem, we have from which
* By the Pythagorean Inequality Theorem, we have from which
* Taking the intersection produces for this case.
* At the obtuse triangle degenerates into a straight line with area at the obtuse triangle degenerates into a right triangle with area Together, we obtain or
* Case (2): The longest side has length so
* By the Triangle Inequality Theorem, we have from which
* By the Pythagorean Inequality Theorem, we have from which
* Taking the intersection produces for this case.
* At the obtuse triangle degenerates into a straight line with area at the obtuse triangle degenerates into a right triangle with area Together, we obtain or
* Answer
* It is possible for noncongruent obtuse triangles to have the same area. Therefore, all such positive real numbers are in exactly one of or Taking the exclusive disjunction, the set of all such is from which
* Solution 3
* We have the diagram below.
* 
* We proceed by taking cases on the angles that can be obtuse, and finding the ranges for that they yield .
* If angle is obtuse, then we have that . This is because is attained at , and the area of the triangle is strictly decreasing as increases beyond . This can be observed from by noting that is decreasing in .
* Then, we note that if is obtuse, we have . This is because we get when , yileding . Then, is decreasing as increases by the same argument as before.
* cannot be obtuse since .
* Now we have the intervals and for the cases where and are obtuse, respectively. We are looking for the that are in exactly one of these intervals, and because , the desired range is giving
* Solution 4
* Note: Archimedes15 Solution which I added an answer here are two cases. Either the and are around an obtuse angle or the and are around an acute triangle. If they are around the obtuse angle, the area of that triangle is as we have and is at most . Note that for the other case, the side lengths around the obtuse angle must be and where we have . Using the same logic as the other case, the area is at most . Square and add and to get the right answer
* Solution 5 (Diagrams)
* For we fix and Without the loss of generality, we consider on only one side of
* As shown below, all locations for at which is an obtuse triangle are indicated in red, excluding the endpoints.
* The region in which is obtuse is determined by construction.
* The region in which is obtuse is determined by the corollaries of the Inscribed Angle Theorem.
* 
* For any fixed value of the height from is fixed. We need obtuse to be unique, so there can only be one possible location for As shown below, all possible locations for are on minor arc including but excluding
* 
* Let the brackets denote areas:
* If then will be minimized (attainable). By the same base and height and the Inscribed Angle Theorem, we have
* If then will be maximized (unattainable). For this right triangle, we have
* Finally, we get from which

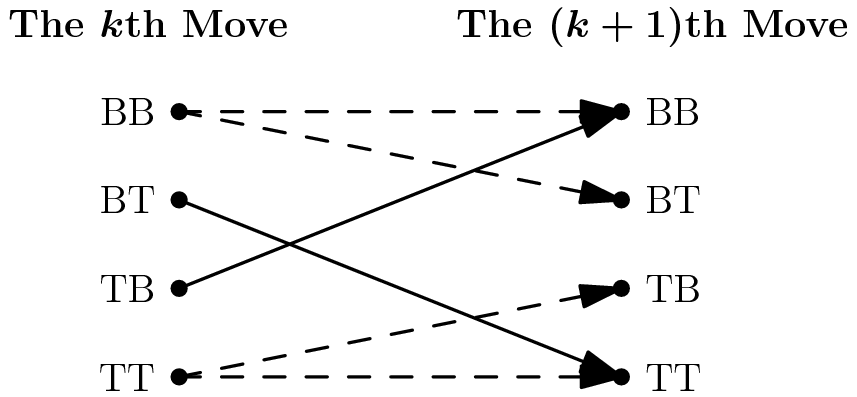
1. 2021 AIME II Problem 6

* For any finite set , let denote the number of elements in . Find the number of ordered pairs such that and are (not necessarily distinct) subsets of that satisfy
* Solution 1
* By PIE, , and after some algebra you see that we need or . WLOG , then for each element there are possibilities, either it is in both and , it is in but not , or it is in neither nor . This gives us possibilities, and we multiply by since it could have also been the other way around. Now we need to subtract the overlaps where , and this case has ways that could happen. It is because each number could be in the subset or it could not be in the subset. So the final answer is .
* Solution 2
* We denote . We denote , , , .
* Therefore, and the intersection of any two out of sets , , , is an empty set. Therefore, is a partition of .
* Following from our definition of , , , we have .
* Therefore, the equation can be equivalently written as
* This equality can be simplified as
* Therefore, we have the following three cases:
  1. and ,
  2. and ,
  3. .
* Next, we analyze each of these cases, separately.
* Case 1: and .
* In this case, to count the number of solutions, we do the complementary counting.
* First, we count the number of solutions that satisfy .
* Hence, each number in falls into exactly one out of these three sets: , , . Following from the rule of product, the number of solutions is .
* Second, we count the number of solutions that satisfy and .
* Hence, each number in falls into exactly one out of these two sets: , . Following from the rule of product, the number of solutions is .
* Therefore, following from the complementary counting, the number of solutions in this case is equal to the number of solutions that satisfy minus the number of solutions that satisfy and , i.e., .
* Case 2: and .
* This case is symmetric to Case 1. Therefore, the number of solutions in this case is the same as the number of solutions in Case 1, i.e., .
* Case 3: and .
* Recall that this is one part of our analysis in Case 1. Hence, the number solutions in this case is .
* By putting all cases together, following from the rule of sum, the total number of solutions is equal to
* Solution 3 (Principle of Inclusion-Exclusion)
* By the Principle of Inclusion-Exclusion (abbreviated as PIE), we have from which we rewrite the given equation as Rearranging and applying Simon’s Favorite Factoring Trick give from which at least one of the following is true:
* Let For each value of we will use PIE to count the ordered pairs
* Suppose There are ways to choose the elements for These elements must also appear in Next, there are ways to add any number of the remaining elements to (Each element has options: in or not in ). There are ordered pairs for Similarly, there are ordered pairs for
* To fix the overcount, we subtract the number of ordered pairs that are counted twice, in which There are such ordered pairs.
* Therefore, there are ordered pairs for
* Two solutions follow from here:
* Solution 3.1 (Binomial Theorem)
* The answer is
* Solution 3.2(Bash)
* The answer is

1. 2021 AIME II Problem 7

* Let and be real numbers that satisfy the system of equations
* There exist relatively prime positive integers and such that Find .
* Solution 1
* From the fourth equation we get substitute this into the third equation and you get . Hence . Solving we get or . From the first and second equation we get , if , substituting we get . If you try solving this you see that this does not have real solutions in , so must be . So . Since , or . If , then the system and does not give you real solutions. So . Since you already know and , so you can solve for and pretty easily and see that . So the answer is .
* Solution 2 (Easy Algebra)
* We can factor out of the last two equations. Therefore, it becomes . Notice this is just , since . We now have and . We then find in terms of , so . We solve for and find that it is either or . We can now try for these two values, and plug the rest into the equation. Thus, we have . We have and we’re done.
* Solution 3 (Easy and Straightforward Algebra)
* can be rewritten as . Hence,
* Rewriting , we get . Substitute and solving, we get, call this Equation 1
* gives . So, , which implies or call this equation 2.
* Substituting Eq 2 in Eq 1 gives,
* Solving this quadratic yields that
* Now we just try these 2 cases.
* For substituting in Equation 1 gives a quadratic in which has roots
* Again trying cases, by letting , we get , Hence We know that , Solving these we get or (doesn’t matter due to symmetry in a,b)
* So, this case yields solutions
* Similarly trying other three cases, we get no more solutions, Hence this is the solution for
* Finally,
* So,
* Solution 4 (Two Variables, Two Equations)
* For simplicity purposes, we number the given equations and in that order.
* Rearranging and solving for we have (5)
* Substituting into and solving for we get (6)
* Substituting and into and simplifying, we rewrite the left side of in terms of and only:
* Let from which Multiplying both sides by rearranging, and factoring give Substituting back and completing the squares produce
* If then combining this with we know that and are the solutions of the quadratic Since the discriminant is negative, neither nor is a real number.
* If then combining this with we know that and are the solutions of the quadratic or from which Substituting into and we obtain and respectively. Together, we have so the answer is

1. 2021 AIME II Problem 8

* An ant makes a sequence of moves on a cube where a move consists of walking from one vertex to an adjacent vertex along an edge of the cube. Initially the ant is at a vertex of the bottom face of the cube and chooses one of the three adjacent vertices to move to as its first move. For all moves after the first move, the ant does not return to its previous vertex, but chooses to move to one of the other two adjacent vertices. All choices are selected at random so that each of the possible moves is equally likely. The probability that after exactly moves that ant is at a vertex of the top face on the cube is , where and are relatively prime positive integers. Find
* Solution 1 (Recursion)
* For all positive integers let
  + be the number of ways to make a sequence of exactly moves, where the last move is from the bottom face to the bottom face.
  + be the number of ways to make a sequence of exactly moves, where the last move is from the bottom face to the top face.
  + be the number of ways to make a sequence of exactly moves, where the last move is from the top face to the bottom face.
  + be the number of ways to make a sequence of exactly moves, where the last move is from the top face to the top face.
* The base case occurs at from which
* Suppose the ant makes exactly moves for some We perform casework on its last move:
* (1)If its last move is from the bottom face to the bottom face, then its next move has
  + way to move from the bottom face to the bottom face.
  + way to move from the bottom face to the top face.
* (2)If its last move is from the bottom face to the top face, then its next move has ways to move from the top face to the top face.
* (3)If its last move is from the top face to the bottom face, then its next move has ways to move from the bottom face to the bottom face. (4)If its last move is from the top face to the top face, then its next move has
  + way to move from the top face to the bottom face.
  + way to move from the top face to the top face.
* Alternatively, this recursion argument is illustrated below, where each dashed arrow indicates way, and each solid arrow indicates ways:
* 
* Therefore, we have the following relationships:
* Using these equations, we recursively fill out the table below:

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| N(,BB) |  |  |  |  |  |  |  |  |
| N(,BT) |  |  |  |  |  |  |  |  |
| N(,TB) |  |  |  |  |  |  |  |  |
| N(,TT) |  |  |  |  |  |  |  |  |
| Total |  |  |  |  |  |  |  |  |

* By the Multiplication Principle, there are ways to make exactly moves. So, we must get for all values of
* Finally, the requested probability is from which the answer is
* Solution 2 (Casework)
* On each move, we can either stay on the level we previously were (stay on the bottom/top) or switch levels (go from top to bottom and vise versa). Since we start on the bottom, ending on the top means that we will have to switch an odd number of times; since we cannot switch twice in a row, over an eight-move period we can either make one or three switches. Furthermore, once we switch to a level we can choose one of two directions of traveling on that level: clockwise or counterclockwise (since we can’t go back to our previous move, our first move on the level after switching determines our direction).
* Case 1: one switch. Our one switch can either happen at the start/end of our moves, or in the middle. There are ways to do this, outlined below. Subcase 1: switch happens at ends. If our first move is a switch, then there are two ways to determine the direction we travel along the top layer. Multiply by 2 to count for symmetry (last move is a switch) so this case yields possibilities. Subcase 2: switch happens in the middle. There are six places for the switch to happen; the switch breaks the sequences of moves into two chains, with each having 2 ways to choose their direction of travel. This case yields possibilities.
* Case 2: three switches. Either two, one, or none of our switches occur at the start/end of our moves. There are ways to do this, outlined below. (Keep in mind we can’t have two switches in a row.) Subcase 1: start and end with a switch. Since our third switch can’t be in moves 2 or 7, there are four ways to place our switch, breaking our sequence into two chains. This case yields possibilities. Subcase 2: one of our switches is at the start/end. WLOG our first move is a switch; moves 2 and 8 cannot be switches. We can choose 2 from any of the remaining 5 moves to be switches, but we have to subtract the 4 illegal cases where the two switches are in a row (3-4, 4-5, 5-6, 6-7). These three switches break our sequence into three chains; accounting for symmetry this case yields possibilities. Subcase 3: all our switches are in the middle. We choose 3 from any of the 6 middle moves to be our switches, but have to subtract the cases where at least two of them are in a row. If at least two switches are in a row, there are five places for the group of 2 and four places for the third switch; however this overcounts the case where all three are in a row, which has 4 possibilities. These three switches break our sequence into four chains, so this case yields possibilities. Our probability is then .
* Solution 3 (Faster Recursion)
* Define to be the probability that after moves, the ant ends up on the level it started on (assuming the first move is a normal move where the ant can stay or move to the opposite level with half chance each). Note and .
* Consider when the ant has moves left. It can either stay on its current level with chance and moves left, or travel to the opposite level with chance and moves left (it must spend another move as it cannot travel back immediately). We then have the recurrence On the first move, the ant can stay on the bottom level with chance and moves left. Or, it can move to the top level with chance and moves left (it has to spend one on the top as it can not return immediately). So the requested probability is .
* Computing we get and , resulting in .

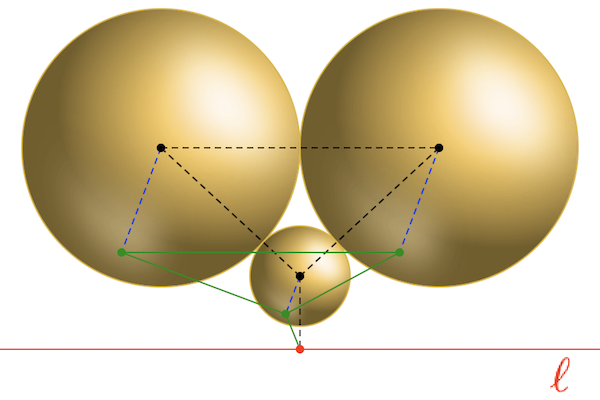
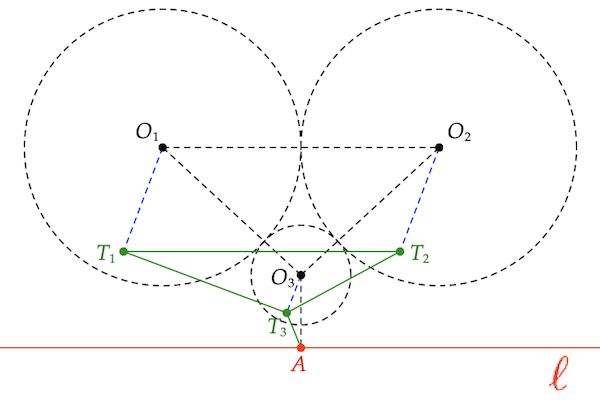
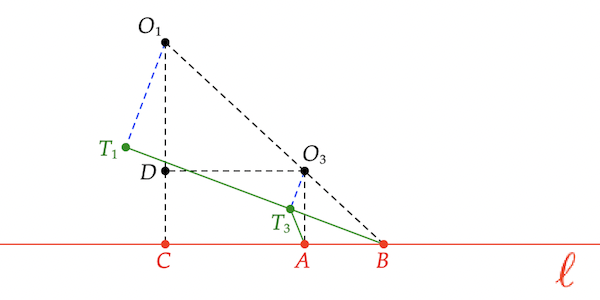
1. 2021 AIME II Problem 9

* Find the number of ordered pairs such that and are positive integers in the set and the greatest common divisor of and is not .
* Solution 1
* This solution refers to the Remarks section.
* By the Euclidean Algorithm, we have We are given that Multiplying both sides by gives
* which implies that must have more factors of than does.
* We construct the following table for the first positive integers:

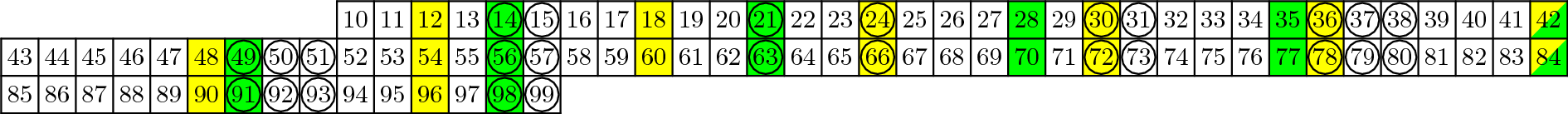
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| --- | --- | --- |
| of Factors of | Numbers | Count |
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* To count the ordered pairs we perform casework on the number of factors of that has:
  + If has factors of then has options and has options. So, this case has ordered pairs.
  + If has factor of then has options and has options. So, this case has ordered pairs.
  + If has factors of then has options and has options. So, this case has ordered pairs.
  + If has factors of then has options and has option. So, this case has ordered pairs.
* Together, the answer is
* Solution 2
* We make use of the (Olympiad Number Theory) lemma that .
* Noting , we have (by difference of squares
* It is now easy to calculate the answer (with casework on ) as .
* Remarks
* Claim 1 (Olympiad Number Theory Lemma)
* If and are positive integers such that then
* There are two proofs to this claim, as shown below.
* Claim 1 Proof 1 (Euclidean Algorithm)
* If then from which the claim is clearly true.
* Otherwise, let without the loss of generality. For all integers and such that the Euclidean Algorithm states that
* We apply this result repeatedly to reduce the larger number:
* Continuing, we have
* from which the proof is complete.
* Claim 1 Proof 2 (Bézout’s Identity)
* Let It follows that and
* By Bézout’s Identity, there exist integers and such that so from which We know that
* Next, we notice that
* Since is a common divisor of and we conclude that from which the proof is complete.
* Claim 2 (GCD Property)
* If and are positive integers such that then
* As and are relatively prime (have no prime divisors in common), this property is intuitive.

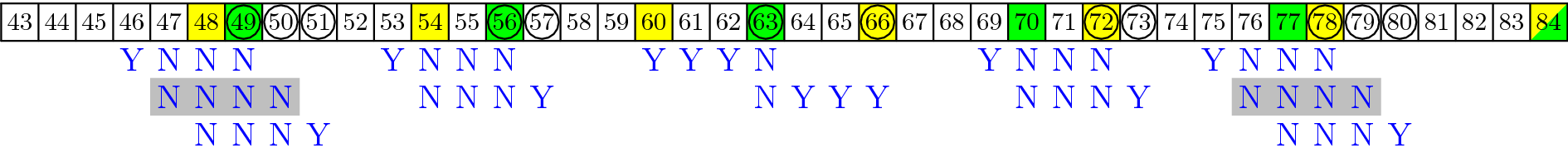
1. 2021 AIME II Problem 10

* Two spheres with radii and one sphere with radius are each externally tangent to the other two spheres and to two different planes and . The intersection of planes and is the line . The distance from line to the point where the sphere with radius is tangent to plane is , where and are relatively prime positive integers. Find .
* 
* Remarks
* Let be the plane that is determined by the centers of the spheres, as shown in the black points. Clearly, the side-lengths of the black dashed triangle are and Plane is tangent to the spheres at the green points. Therefore, the blue dashed line segments are the radii of the spheres.
* We can conclude all of the following:
* The four black dashed line segments all lie in plane
* The four green solid line segments all lie in plane
* By symmetry, since planes and are reflections of each other about plane the three planes are concurrent to line
* The red point is the foot of the perpendicular from the smallest sphere’s center to line
* Solution 1 (Similar Triangles and Pythagorean Theorem)
* This solution refers to the Diagram section.
* As shown below, let be the centers of the spheres (where sphere is the smallest) and be their respective points of tangency to plane Suppose is the foot of the perpendicular from to line so is the perpendicular bisector of We wish to find
* 
* As the intersection of planes and is line we know that both and must intersect line Furthermore, since and it follows that from which and are coplanar.
* Now, we focus on cross-sections and
* In the three-dimensional space, the intersection of a line and a plane must be exactly one of the empty set, a point, or a line.
* Clearly, cross-section intersects line at exactly one point. Let the intersection of and line be which must also be the intersection of and line
* In cross-section let be the foot of the perpendicular from to line and be the foot of the perpendicular from to
* We have the following diagram:
* 
* In cross-section since as discussed, we obtain by AA, with the ratio of similitude Therefore, we get or
* In cross-section note that and Applying the Pythagorean Theorem to right we have Moreover, since and we obtain so that by AA, with the ratio of similitude Therefore, we get or
* Finally, note that and Since is a rectangle, we have Applying the Pythagorean Theorem to right gives from which the answer is
* Solution 2
* The centers of the three spheres form a -- triangle. Consider the points at which the plane is tangent to the two bigger spheres; the line segment connecting these two points should be parallel to the side of this triangle. Take its midpoint , which is away from the midpoint of the side, and connect these two midpoints.
* Now consider the point at which the plane is tangent to the small sphere, and connect with the small sphere’s tangent point . Extend through until it hits the ray from through the center of the small sphere (convince yourself that these two intersect). Call this intersection , the center of the small sphere , we want to find .
* By Pythagoras, , and we know that and . We know that and must be parallel, using ratios we realize that . Apply the Pythagorean theorem to , , so .
* Solution 3 (Coordinates Bash)
* Let’s try to see some symmetry. We can use an -plane to plot where the circles are. The two large spheres are externally tangent, so we’ll make them at and . The center of the little sphere would be since we don’t know how much the little sphere will be “pushed” down. We use the 3D distance formula to find that (since wouldn’t make sense). Now, we draw a line through the little sphere and the origin. It also intersects because of the symmetry we created.
* lies on the plane too, so these two lines must intersect. The point at where it intersects is . We can use the distance formula again to find that . Therefore, they intersect at . Since the little circle’s -coordinate is and the intersection point’s -coordinate is , we get . Therefore, our answer to this problem is .

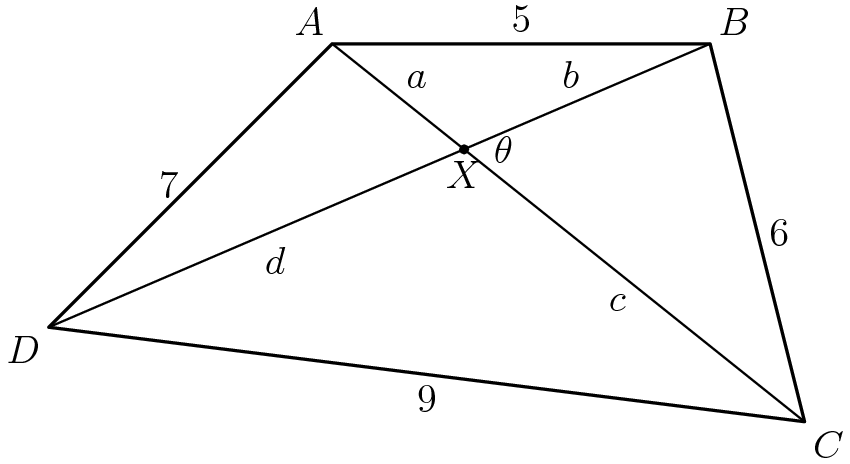
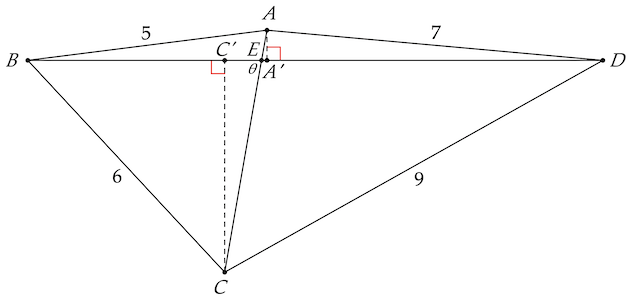
1. 2021 AIME II PProblem 11

* A teacher was leading a class of four perfectly logical students. The teacher chose a set of four integers and gave a different number in to each student. Then the teacher announced to the class that the numbers in were four consecutive two-digit positive integers, that some number in was divisible by , and a different number in was divisible by . The teacher then asked if any of the students could deduce what is, but in unison, all of the students replied no.
* However, upon hearing that all four students replied no, each student was able to determine the elements of . Find the sum of all possible values of the greatest element of .
* Solution 1 Note that It is clear that and otherwise the three other elements in are divisible by neither nor
* In the table below, the multiples of are colored in yellow, and the multiples of are colored in green. By the least common multiple, we obtain cycles: If is a possible maximum value of then must be another possible maximum value of and vice versa. By observations, we circle all possible maximum values of
* 
* From the second row of the table above, we perform casework on the possible maximum value of

|  |  |  |  |
| --- | --- | --- | --- |
| Max Value | S | Valid? | Reasoning/Conclusion |
|  |  |  | The student who gets will reply yes. |
|  |  |  | Another possibility is |
|  |  |  | The student who gets will reply yes. |
|  |  |  | The student who gets will reply yes. |
|  |  |  | The student who gets will reply yes. |
|  |  |  | The students who get will reply yes. |
|  |  |  | The students who get will reply yes. |
|  |  |  | The student who gets will reply yes. |
|  |  |  | The student who gets will reply yes. |
|  |  |  | The student who gets will reply yes. |
|  |  |  | Another possibility is . |
|  |  |  | The student who gets will reply yes. |

* Finally, all possibilities for are and from which the answer is
* Remarks
  + Alternatively, we can reconstruct the second table in this solution as follows, where Y and N denote the replies of “yes” and “no”, respectively. Notice that this table has some kind of symmetry!
* 
  + As a confirmation, we can verify that each student will be able to deduce what is upon hearing the four replies of “no” in unison. For example, if then all students will know that no one gets or otherwise that student will reply yes (as discussed). Therefore, all students will conclude that has only one possibility.

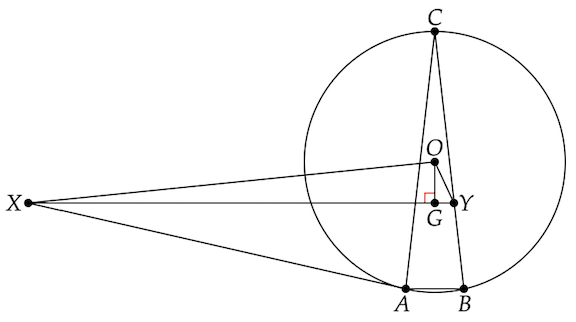
1. 2021 AIME II Problem 12

* A convex quadrilateral has area and side lengths and in that order. Denote by the measure of the acute angle formed by the diagonals of the quadrilateral. Then can be written in the form , where and are relatively prime positive integers. Find .
* Solution 1 (Sines and Cosines) Since we are asked to find , we can find and separately and use their values to get . We can start by drawing a diagram. Let the vertices of the quadrilateral be , , , and . Let , , , and . Let , , , and . We know that is the acute angle formed between the intersection of the diagonals and .
* 
* We are given that the area of quadrilateral is . We can express this area using the areas of triangles , , , and . Since we want to find and , we can represent these areas using as follows:
* We know that . Therefore it follows that:
* From here we see that . Now we need to find . Using the Law of Cosines on each of the four smaller triangles, we get following equations:
* From here we see that .
* Since we have figured out and , we can calculate :
* Therefore our answer is .
* Steven Chen (www.professorchenedu.com)
* Solution 2 (Right Triangles)
* In convex quadrilateral let and Let and be the feet of the perpendiculars from and respectively, to We obtain the following diagram:
* 
* Let and We apply the Pythagorean Theorem to right triangles and respectively:
* Let the brackets denote areas. We get
* We subtract from $(1)+(3):
* From right triangles and we have It follows that
* Finally, we get:
* from which the answer is
* Solution 3
* Let , , , be the vertices of the quadrilateral, , , , be the lengths of the sides of , and and be the lengths of the diagonals of .
* By Bretschneider’s formula, . Thus, . Also, ; solving for yields .
* Since is acute, is positive, so .
* Solving for yields , for a final answer of .

1. 2021 AIME II Problem 13

* Find the least positive integer for which is a multiple of .
* Solution 1
* divides this expression iff and both divide it. It should be fairly obvious that ; so we may break up the initial condition into two sub-conditions.
  1. . Notice that the square of any odd integer is modulo (proof by plugging in into modulo ), so the LHS of this expression goes , while the RHS goes . The cycle length of the LHS is , RHS is , so the cycle length of the solution is . Indeed, the that solve this congruence are exactly those such that .
  2. . This is extremely computationally intensive if you try to calculate all , so instead, we take a divide-and-conquer approach. In order for this expression to be true, is necessary; it shouldn’t take too long for you to go through the possible LHS-RHS combinations and convince yourself that or .
* With this in mind we consider . By the Generalized Fermat’s Little Theorem, , but we already have modulo . Our calculation is greatly simplified. The LHS cycle length is , RHS cycle length is , the lcm is , in this step we need to test all the numbers between to that or . In the case that , the RHS goes , and we need ; clearly . In the case that , by a similar argument, .
* In the final step, we need to calculate and modulo :
* Note that ; because we get .
* Note that is . We have
* This time, LHS cycle is , RHS cycle is , so we need to figure out modulo . It should be .
* Put everything together. By the second subcondition, the only candidates less than are . Apply the first subcondition, is the desired answer.
* Solution 2
* We have that , or and by CRT. It is easy to check don’t work, so we have that . Then, and , so we just have and . Let us consider both of these congruences separately.
* First, we look at . By Euler’s Totient Theorem (ETT), we have , so . On the RHS of the congruence, the possible values of are all nonnegative integers less than and on the RHS the only possible values are and . However, for to be we must have , a contradiction. So, the only possible values of are when .
* Now we look at . Plugging in , we get . Note, for to be satisfied, we must have and . Since as , we have . Then, . Now, we get . Using the fact that , we get . The inverse of modulo is obviously , so , so . Plugging in , we get .
* Now, we are finally ready to plug into the congruence modulo . Plugging in, we get . By ETT, we get , so . Then, . Plugging this in, we get , implying the smallest value of is simply .
* ~rocketsri
* Solution 3 (Chinese Remainder Theorem and Binomial Theorem)
* We wish to find the least positive integer for which Rearranging gives
* Applying the Chinese Remainder Theorem, we get the following system of congruences:
* It is clear that from which we simplify to
* (1)
* (2)
* We solve each congruence separately:
  + For quick inspections produce that are congruent to modulo respectively. More generally, if is odd, and if is even. As is always odd (so is ), we must have
  + That is, for some nonnegative integer
  + For we substitute the result from and simplify:
  + Note that and so we apply the Binomial Theorem to the left side:
  + Since it follows that
  + That is, for some nonnegative integer
  + Substituting this back into we get
  + As is a multiple of it has at least three factors of Since contributes one factor, contributes at least two factors, or must be a multiple of Therefore, the least such nonnegative integer is
* Finally, combining the two results from above (bolded) generates the least such positive integer at
* Solution 4 (Minimal Computation)
* Note that and so is periodic every . Easy to check that only satisfy. Let . Note that by binomial theorem,
* So we have
* Combining with gives that is in the form of , . Note that since
* Easy to check that only

1. 2021 AIME II Problem 14

* Let be an acute triangle with circumcenter and centroid . Let be the intersection of the line tangent to the circumcircle of at and the line perpendicular to at . Let be the intersection of lines and . Given that the measures of and are in the ratio the degree measure of can be written as where and are relatively prime positive integers. Find .
* 
* Solution 1
* In this solution, all angle measures are in degrees.
* Let be the midpoint of so that and are collinear. Let and
* Note that:
  + Since quadrilateral is cyclic by the Converse of the Inscribed Angle Theorem. It follows that as they share the same intercepted arc
  + Since quadrilateral is cyclic by the supplementary opposite angles. It follows that as they share the same intercepted arc
* Together, we conclude that by AA, so
* Next, we express in terms of By angle addition, we have
* Substituting back gives from which
* For the sum of the interior angles of we get
* Finally, we obtain from which the answer is
* Solution 2
* Let be the midpoint of . Because , and are cyclic, so is the center of the spiral similarity sending to , and .
* Because , it’s easy to get from here.
* Solution 3 (Easy and Simple)
* Firstly, let be the midpoint of . Then, . Now, note that since , quadrilateral is cyclic. Also, because , is also cyclic.
* Now, we define some variables: let be the constant such that and .
* Also, let and (due to the fact that and are cyclic).
* Then,
* Now, because is tangent to the circumcircle at , , and .
* Finally, notice that .
* Then,
* Thus,
* and
* However, from before, , so . To finish the problem, we simply compute
* so our final answer is .
* Solution 4 (Guessing in the Last 3 Minutes, Unreliable)
* Notice that looks isosceles, so we assume it’s isosceles. Then, let and Taking the sum of the angles in the triangle gives so so the answer is

1. 2021 AIME II Problem 15

* Let and be functions satisfying
* and
* for positive integers . Find the least positive integer such that .
* Solution 1
* Consider what happens when we try to calculate where n is not a square. If for (positive) integer k, recursively calculating the value of the function gives us . Note that this formula also returns the correct value when , but not when . Thus for .
* If , returns the same value as . This is because the recursion once again stops at . We seek a case in which , so obviously this is not what we want. We want to have a different parity, or have the same parity. When this is the case, instead returns .
* Write , which simplifies to . Notice that we want the expression to be divisible by 3; as a result, . We also want n to be strictly greater than , so . The LHS expression is always even (why?), so to ensure that k and n share the same parity, k should be even. Then the least k that satisfies these requirements is , giving .
* Indeed - if we check our answer, it works. Therefore, the answer is .
* Solution 2 (Four Variables)
* We consider and separately:
  + We restrict in which for some positive integer or
  + for some integer such that
  + By recursion, we get
  + If and have the same parity, then we get by a similar process from This contradicts the precondition Therefore, and must have different parities, from which and must have the same parity.
  + It follows that or
  + for some integer such that By recursion, we get
* Answer
* By and we have
* From and equating the expressions for gives Solving for produces
* We substitute into then simplify, cross-multiply, and rearrange:
* Since we know that must be divisible by and must be divisible by
* Recall that the restrictions on and are and respectively. Substituting into either inequality gives Combining all these results produces
* To minimize in either or we minimize so we minimize and in From and we construct the following table:

|  |  |  |  |
| --- | --- | --- | --- |
| * p | * q | * k | * Satisfies (8) |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

* Finally, we have Substituting this result into either or generates
* Remark
* We can verify that
* Solution 3
* Since isn’t a perfect square, let with . If is odd, then . If is even, then
* from which
* Since is even, is even. Since , the smallest is which produces the smallest :
* Solution 4 (Quadratics With Two Variables)
* To begin, note that if is a perfect square, , so , so we must look at values of that are not perfect squares (what a surprise). First, let the distance between and the first perfect square greater than or equal to it be , making the values of and integers. Using this notation, we see than , giving us a formula for the numerator of our ratio. However, since the function of does not add one to the previous inputs in the function until a perfect square is achieved, but adds values of two, we can not achieve the value of in unless is an even number. However, this is impossible, since if was an even number, , giving a ratio of one. Thus, must be an odd number.
* Thus, since must be an odd number, regardless of whether is even or odd, to get an integral value in , we must get to the next perfect square after . To make matters easier, let . Thus, in , we want to achieve .
* Expanding and substituting in the fact that yields:
* Thus, we must add the quantity to to achieve a integral value in the function . Thus.
* However, note that the quantity within the square root is just , and so:
* Thus,
* Since we want this quantity to equal , we can set the above equation equal to this number and collect all the variables to one side to achieve
* Substituting back in that , and then separating variables and squaring yields that
* Now, if we treat as a constant, we can use the quadratic formula in respect to to get an equation for in terms of (without all the squares). Doing so yields
* Now, since and are integers, we want the quantity within the square root to be a perfect square. Note that . Thus, assume that the quantity within the root is equal to the perfect square, . Thus, after using a difference of squares, we have[(m-55)(m+55)=900n]Since we want to be an integer, we know that the should be divisible by five, so, let’s assume that we should have divisible by five. If so, the quantity must be divisible by five, meaning that leaves a remainder of one when divided by 5 (if the reader knows LaTeX well enough to write this as a modulo argument, please go ahead and do so!).
* Thus, we see that to achieve integers and that could potentially satisfy the problem statement, we must try the values of congruent to one modulo five. However, if we recall a statement made earlier in the problem, we see that we can skip all even values of produced by this modulo argument.
* Also, note that won’t work, as they are too small, and will give an erroneous value for . After trying , we see that will give a value of , which yields , which, if plugged in to for our equations of and , will yield the desired ratio, and we’re done.
* Side Note: If any part of this solution is not rigorous, or too vague, please label it in the margin with “needs proof”. If you can prove it, please add a lemma to the solution doing so :)
* Solution 5 (Basic Substitutions)
* First of all, if is a perfect square, and their quotient is So, for the rest of this solution, assume is not a perfect square.
* Let be the smallest perfect square greater than and let be the smallest perfect square greater than with the same parity as and note that either or Notice that
* With a bit of inspection, it becomes clear that and
* If and have the same parity, we get so and their quotient is So, for the rest of this solution, we let and have opposite parity. We have two cases to consider.
* Case 1: is odd and is even
* Here, we get for some positive integer Then, We let for some positive integer so and
* We set cross multiply, and rearrange to get Since and are integers, the LHS will always be even and the RHS will always be odd, and thus, this case yields no solutions.
* Case 2: is even and is odd
* Here, we get for some positive ineger Then, We let for some positive integer so and
* We set cross multiply, and rearrange to get or Since and are integers, must be a multiple of Some possible solutions for with the least and are and
* We wish to minimize since One thing to keep in mind is the initial assumption
* The pair gives and But is clearly false, so we discard this case.
* The pair gives and which is a perfect square and therefore can be discarded.
* The pair gives and which is between and so it is our smallest solution.
* So, is the correct answer.